MICROWAVE BACKGROUND: PHYSICS OF THE SPECTRUM

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ABSTRACT. The purpose of this first lecture on the μ-wave background radiation is to develop some of the physics related to the black-body spectrum. This includes time-scales for important processes, decoupling/recombination calculations, residual ionization, the phenomenology of spectral distortions, and some discussion of the current state of spectral observations. Temperature fluctuations will not be examined here but will be covered in later lectures.

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1. Introduction

The present universe is filled with leftovers from earlier times, and the hot big bang insures that, to a first approximation, these leftovers have a relatively simple structure, having started in a hot thermal state. These leftovers are selectively cooled and ”frozen-out” of the primordial gas, in successive stages of evolution.

We have seen how various nuclear burning processes freeze-out at temperatures $T \lesssim 1\,\mathrm{MeV}$, leading to the production of light nuclei, notably helium and deuterium.

A similar story applies to the ionization density of the Universe. When the temperature drops, the ionization density decreases sharply and then freezes out. Because the residual ionization is quite low, the Universe becomes transparent to radiation. We would then expect to see a stream of photons coming from all directions in the sky, emitted from a (somewhat fuzzy) sphere at cosmological distance. So a relic radiation background is a necessary consequence of a hot big bang. Of course, the nature of the radiation spectrum remains to be determined. In this lecture, we will verify these statements and show how everything works out to produce a relic black-body spectrum.

Recall that a thermal spectrum of photons (a black-body spectrum) has an evolution in an expanding universe which is completely described by the temperature scaling

$$\frac{T(t_1)}{T(t_2)} = \frac{a(t_2)}{a(t_1)} = \frac{1+z(t_1)}{1+z(t_2)},$$

where $a(t)$ is the scale factor and $z(t)$ is the redshift. This relation was understood by Tolman in 1936, although there is no indication that the possibility of a radiation background was considered at that time.

This scaling relation holds as long as the radiation is either thermodynamically dominant or uncoupled. In particular, an uncoupled radiation component simply redshifts indefinitely into the future.

2. A Historical Guess for the Background Temperature

The first serious consideration of the hot big bang was the investigation of light element nucleosynthesis by Gamow (1946, 1948) and Alpher and Herman (1948). Nucleosynthesis constrains the baryon density, and Alpher and Herman pointed out that this can be used as a chronometer to determine the amount of redshift which must have occured since nucleosynthesis. The argument works as follows.

First we boil down all of nucleosynthesis to the following observation. The most important reaction is the first step,

$$p + n \longleftrightarrow d + \gamma.$$

This step is important both because it is first and because the relatively low binding energy of the deuteron makes photo-dissociation by the reverse reaction a key bottleneck.

Gamow's basic observation about nucleosynthesis was that this reaction must be effective, but not too effective, at the time of nucleosynthesis. Clearly it must have some effect, or nothing happens. If it is too effective, then the light elements burn away. So the Gamow criterion is

$$\langle \sigma v \rangle n_B(t_{\mathrm{nuc}}) t_{\mathrm{nuc}} \approx 1,$$

at the temperature where deuterium production can really begin, $T_{\mathrm{nuc}} \approx 10^9\,\mathrm{K} \approx 0.1\,\mathrm{MeV}$. Recall that the Einstein equation in a radiation-dominated universe gives

$$\begin{aligned}
H(t)^2 = \frac{8\pi}{3} G\rho_{\mathrm{RAD}} &= \frac{8\pi}{3}\frac{\pi^2}{30} g_* \frac{T^4}{m_{\mathrm{Pl}}^2}, \\
\Longrightarrow \qquad t^{-1} \simeq H &= 1.7 g_*^{1/2} \frac{T^2}{m_{\mathrm{Pl}}}, \\
\Longrightarrow \qquad t_{\mathrm{nuc}}^{-1} &\simeq 4 \times 10^{-24}\,\mathrm{MeV}, \\
&\simeq 5 \times 10^{-3}\,\mathrm{sec}^{-1},
\end{aligned}$$

The averaged cross-section at $0.1\,\mathrm{MeV}$ is $\langle \sigma v \rangle \approx 10^{-31}\,\mathrm{cm}^2$. So we determine the baryon density as

$$\begin{aligned}
n_B(t_{\mathrm{nuc}}) &\approx t_{\mathrm{nuc}}^{-1} \langle \sigma v \rangle^{-1} \\
&\approx 2 \times 10^{18}\,\mathrm{cm}^{-3}.
\end{aligned}$$

This completes our poor-man's nucleosynthesis calculation. If we now use an estimate for the current baryon density, we can obtain the expansion factor since nucleosynthesis, and therefore obtain the redshift. From estimates of the mass to luminosity ratio of the Universe we know, roughly,

$$n_B(t_{\mathrm{now}}) \approx 5 \times 10^{-8}\,\mathrm{cm}^{-3}.$$

Use the relation which is equivalent to number conservation,

$$\frac{T_\gamma(t_{\mathrm{now}})}{T(t_{\mathrm{nuc}})} = \frac{a(t_{\mathrm{nuc}})}{a(t_{\mathrm{now}})} = \left(\frac{n_B(t_{\mathrm{now}})}{n_B(t_{\mathrm{nuc}})} \right)^{1/3},$$

and so $T_\gamma(t_{\mathrm{now}}) \approx 5\mathrm{K}$. This is a surprisingly (suspiciously) good number. It is also coincidentally the number obtained by Alpher and Herman after a more careful estimate. Note that we assumed part of what we want to show later, which is that the radiation is either thermodynamically dominant or uncoupled; otherwise we could not have used the $T \propto a^{-1}$ scaling.

Perhaps the most remarkable aspect of this estimate, from a modern perspective, is the fact that it essentially vanished into obscurity for almost twenty years.

Theory can probably do a little better now than Alpher and Herman. But it makes more sense to measure the current radiation temperature and use it as a fundamental input for cosmology. This measured temperature is a very well-determined quantity, $T_\gamma(t_{\rm now}) = 2.728 \pm 0.002\,{\rm K}$, a global average.

3. Thermal Ionization History

Leaving nucleosynthesis in the past, we eventually come to a time when the temperature was about 1 eV. This occurs at an approximate age

$$\begin{aligned} t_{1\,{\rm eV}} &\simeq 200\,{\rm sec}\left(\frac{0.1\,{\rm MeV}}{1\,{\rm eV}}\right)^2 \\ &\simeq 2\times 10^{12}\,{\rm sec} \simeq 10^5\,{\rm yr}. \end{aligned}$$

In fact, this is not quite right because we have assumed radiation domination. It turns out that, also around this time, the matter and radiation energy densities are becoming comparable. The form for the Hubble expansion which displays this crossover from radiation to matter domination is

$$H^{-1} = \begin{cases} 1.5\times 10^{12}\,{\rm sec}\left(\frac{1\,{\rm eV}}{T_\gamma}\right)^2, & t < t_{\rm EQ} \\ 1.1\times 10^{12}\,{\rm sec}\left(\frac{1\,{\rm eV}}{T_\gamma}\right)^{3/2}\left(\Omega_0 h^2\right)^{-1/2}, & t > t_{\rm EQ}. \end{cases}$$

Matter-radiation equality occurs at a temperature

$$T_{\rm EQ} \simeq 5.5\,{\rm eV}\,\Omega_0 h^2 \left(\frac{2.75\,{\rm K}}{T_\gamma(t_{\rm now})}\right)^3.$$

As discussed above, we expect the ionization to drop suddenly when the temperature falls into the eV range, as ions and electrons combine to form neutral atoms. This is a fairly naive picture of what is happening because we really must check the various thermalization conditions that are implicitly assumed. But let's pursue this picture far enough to obtain an estimate for the temperature at which the ionization fraction becomes small.

We will assume that the Universe consists purely of protons and electrons, and that the number densities of these species are given by equilibrium values. Recall that equilibrium number densities for non-relativistic particles have the form

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2}\exp\left(\frac{\mu - m}{T}\right).$$

Combining the number densities for protons, electrons and neutral hydrogen atoms we get

$$\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H}\left(\frac{T}{2\pi}\frac{m_e m_p}{m_H}\right)^{3/2}\exp\left(\frac{\mu_e + \mu_p - \mu_H + m_H - m_e - m_p}{T}\right).$$

By assumption the reaction $p + e \longleftrightarrow H + \gamma$ is in equilibrium. So $\mu_e + \mu_p = \mu_H$. Also, $m_p \simeq m_H$, and $m_H - m_e - m_p = -B_H \simeq -13.6\,\mathrm{eV}$. So

$$\frac{n_e n_p}{n_H} \simeq \frac{g_e g_p}{g_H}\left(\frac{m_e T}{2\pi}\right)^{3/2}\exp\left(-\frac{B_H}{T}\right).$$

Define $n_H = (1 - x)n_B$; $n_e = n_p = x n_B$. Also note that $g_e = g_p = 2$. Less obvious is the choice $g_H = 4$; we could treat the hyperfine states separately, but the fine and hyperfine splitting is so low that we obtain the same result if we lump them together at the start. So we have

$$\frac{x^2}{1 - x} \simeq \left(\frac{m_e T}{2\pi}\right)^{3/2}\frac{1}{n_B(T)}\exp\left(-\frac{B_H}{T}\right).$$

This is the Saha equation, which gives the ionization fraction for our thermal plasma of protons and electrons.

Recall the definition $\eta \equiv n_B/n_\gamma$. Also, the number density of thermal photons is given by

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}T_\gamma^3.$$

So we can write

$$\frac{x^2}{1 - x} \simeq \frac{\pi^2}{2\zeta(3)}\frac{1}{(2\pi)^{3/2}}\eta^{-1}\exp\left(-\frac{B_H}{T}\right)\left(\frac{m_e}{T}\right)^{3/2}.$$

Notice the large factor of η^{-1}. This factor has the same origin as it did in similar expressions for nucleosynthesis; the Universe has so many more photons than baryons that entropy wins over energy until the temperature becomes quite low.

Using the measured background radiation temperature to fix the ratio at present times, we can write $\eta \simeq 2.68 \times 10^{-8}\Omega_B h^2$. So now we can see the large factor in the Saha equation explicitly,

$$\frac{x^2}{1 - x} \simeq 1.77 \times 10^{17}\left(\frac{\Omega_B h^2}{0.02}\right)^{-1}\left(\frac{1\,\mathrm{eV}}{T}\right)^{3/2}\exp\left(-\frac{B_H}{T}\right).$$

Take $\Omega_B h^2 = 0.02$ for example. Then we obtain the results shown in the table.

x	T (eV)
0.92	0.35
0.65	0.33
0.14	0.30
0.031	0.28
0.0018	0.25
2.4×10^{-6}	0.20

So the thermal prediction for the ionization drops rapidly at temperatures close to 0.28 eV; the dependence of this temperature on $\Omega_B h^2$ is logarithmically weak. The redshift at this temperature is $z \simeq 1300$. But note that the redshift range over which this ionization drop occurs is not negligible; $\delta z \approx 100$.

This process of loss of ionization and formation of neutral hydrogen is called recombination. The decoupling of radiation is logically distinct from recombination, since it depends on the mean free path for photons; however, we will continue to think of these events as simultaneous in this section.

If decoupling happened instantaneously, then clearly the relic background spectrum would be thermal. But we see that recombination is not instantaneous, and one can guess, correctly, that decoupling is not instantaneous either. Therefore the temperature of the matter and the temperature of the radiation are evolving through this time, and the source of the relic radiation is smeared throughout this window in time.

At this point we should begin to worry a little. The matter is non-relativistic, so we might expect decoupling to smear the radiation spectrum in some way which is not consistent with a black-body spectrum. Even worse, the Universe is probably matter-dominated at this time. What is going on here?

The key is to realize two things. Firstly, there is a difference between energy dominance and thermodynamic dominance. Secondly, we know that the photon bath has a very large heat capacity compared to the matter, and it carries a large amount of entropy, so we expect it to dominate thermodynamically. We can see how this happens in a simple model. The calculation for this model is instructive. Consider a system consisting of a thermal radiation field in equilibrium with a non-relativistic gas,

$$\rho = n_M m + (\gamma - 1)^{-1} n_M T + \alpha T^4,$$
$$p = n_M T + \frac{1}{3}\alpha T^4.$$

The equation of energy conservation, $\nabla_a T_{ab} = 0$, is

$$\frac{dp}{dt} a(t)^3 = \frac{d}{dt}\left(a(t)^3(\rho + p)\right).$$

$$\implies \qquad a(t)^3 d\rho + 3a(t)^2(\rho + p)da = 0,$$
$$\implies \qquad \frac{d}{da}\left(\rho a^3\right) + 3pa^2 = 0,$$
$$\implies \qquad \frac{d}{da}\left(n_M m a^3 + (\gamma-1)^{-1} n_M T a^3 + \alpha a^3 T^4\right) = -3n_M T a^2 - \alpha a^2 T^4.$$

By number conservation, $n_M a^3$ is constant. So

$$(\gamma-1)^{-1} n_M a^3 \frac{da}{dT} + \alpha\left(4a^3T^3\frac{dT}{da} + 3a^2T^4\right) = -3n_M a^3 \frac{T}{a} - \alpha a^2 T^4,$$
$$\implies \qquad \frac{a}{T}\frac{dT}{da}\left((\gamma-1)^{-1} n_M a^3 + 4\alpha a^3 T^3\right) = -3n_M a^3 - 4\alpha a^3 T^3,$$
$$\implies \qquad \frac{a}{T}\frac{dT}{da} = -\left(\frac{1 + \frac{4}{3}\alpha\frac{T^3}{n_M}}{\frac{1}{3}(\gamma-1)^{-1} + \frac{4}{3}\alpha\frac{T^3}{n_M}}\right).$$

But

$$\frac{3}{4}\frac{n_M}{\alpha T^3} = \frac{3}{4}\frac{30\zeta(3)}{\pi^4}\frac{n_M}{n_\gamma} = c\eta; \quad c \simeq 2.03.$$

So

$$\frac{a}{T}\frac{dT}{da} = -\left(\frac{1 + (c\eta)^{-1}}{\frac{1}{3}(\gamma-1)^{-1} + (c\eta)^{-1}}\right).$$

This is the result we need. When η is very small we have

$$\frac{d\ln T}{d\ln a} = -1 \quad \implies \qquad T(t)a(t) = \text{constant}.$$

This is a demonstration of a general fact that we have mentioned before. If the matter is in thermal contact with the radiation, and if the number density of the matter is relatively small, so that the entropy is dominated by the radiation, then the thermodynamics is dominated by the radiation. This is completely independent of which component may happen to dominate energetically.

Simply put, the photons overwhelm the matter by force of numbers, and the matter fails to have a thermodynamic will of its own; its temperature scaling is locked to the radiation temperature scaling. Therefore, under these circumstances, *it does not matter* how decoupling occurs. As long as the charged particles from which the photons scatter have the same temperature scaling as the radiation, the radiation spectrum at any time downstream will be precisely black-body.

Because we know that η is small for our Universe, this conclusion will fail to hold only if the matter loses thermal contact with the radiation. When this happens, the assumption that the temperatures are the same is violated, and we obtain the scaling that we expect for matter alone,

$$\frac{d\ln T}{d\ln a} = -3(\gamma - 1) \quad \Longrightarrow \qquad T(t)a(t)^{3(\gamma-1)} = \text{constant},$$

or $T(t)a(t)^2 = \text{constant}$ for the usual $\gamma = 5/3$.

So now the question arises as to whether or not the matter is in thermal contact with the radiation. No equilibrium arguments suffice to answer this, so we must move on to dynamical estimates.

4. Thermal Coupling of Matter and Radiation

We begin this discussion by cataloguing the important processes in the plasma. Of course, this will be a simplified picture, but we will be able to demonstrate what we need from what is given here.

4.1. **Coulomb collisions.** First we note that the charged particles manage to remain thermal, whether coupled to radiation or not. To check this, note that Coulomb scattering gives a time-scale

$$t_{\text{Coulomb}} = \frac{1}{n_e\langle\sigma v\rangle} \approx \frac{1}{xn_B}\left(\frac{m}{T}\right)^{1/2}\left(\frac{\alpha_{em}}{T}\right)^{-2}.$$

The absolutely longest time-scale will be for proton-proton collisions because of the mass dependence, and for these we get

$$t_{\text{pp}} \approx 5000\,\text{sec}\left(\frac{1\,\text{eV}}{T}\right)^{3/2}\frac{1}{x}\frac{1}{\Omega_B h^2}.$$

In fact, the true time-scale is even shorter because protons will equilibrate by scattering off electrons. Also, I left a large ”Coulomb log” out of the cross-section, which depends on screening effects. In any case, clearly this time-scale is much shorter than H^{-1} at this time. So the charged particles have no trouble remaining thermal for a long time, as long as the ionization fraction is not very small.

4.2. **Thomson scattering.** This is the non-relativistic limit for $e + \gamma \longrightarrow e + \gamma$.

$$\begin{aligned}\sigma_T &= \frac{8\pi}{3}\left(\frac{\alpha_{em}}{m_e}\right)^2, \\ &\simeq 0.0017\,\text{MeV}^{-2} \simeq 6.6\times10^{-25}\,\text{cm}^2.\end{aligned}$$

We want to be careful about how we treat the matter and the radiation. There are really two time-scales of interest. One is the mean free time for photons in a background electron density; the other is the mean free time for electrons in the background radiation. The mean free time for photons is

$$t_{\mathrm{T},\gamma} \simeq 3.1 \times 10^9 \sec \left(\frac{1\,\mathrm{eV}}{T}\right)^3 \frac{1}{x}\frac{0.02}{\Omega_B h^2}.$$

The mean-free time for electrons is

$$t_{\mathrm{T,e}} \simeq 1.7 \sec \left(\frac{1\,\mathrm{eV}}{T_\gamma}\right)^3.$$

Both of these time-scales are short compared to the Hubble time. Therefore Thomson scattering is effective throughout this era. But we have a problem here. Thomson scattering is large because it involves no energy transfer; energy transfer can occur only if we allow relativistic effects. Without energy transfer, Thomson scattering simply randomizes the directions of photons but has no effect on their energy distribution.

4.3. **Compton scattering.** Compton scattering is the fully relativistic result for which Thomson scattering is the non-relativistic limit. Energy transfer occurs at order $(v/c)^2$ or T/m, where T is the temperature of the electron bath. Therefore

$$\begin{aligned} t_{\mathrm{Compton},\gamma} &\simeq t_{\mathrm{T},\gamma}\frac{m_e}{T_e} \simeq 1.6 \times 10^{15} \sec \left(\frac{1\,\mathrm{eV}}{T_\gamma}\right)^4 \frac{1}{x}\left(\frac{0.02}{\Omega_B h^2}\right)\frac{T_\gamma}{T_e}, \\ t_{\mathrm{Compton,e}} &\simeq t_{\mathrm{T,e}}\frac{m_e}{T_e} \simeq 8.6 \times 10^5 \sec \left(\frac{1\,\mathrm{eV}}{T_\gamma}\right)^4 \frac{T_\gamma}{T_e}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\Gamma_{\mathrm{Compton},\gamma}}{H} &\simeq x\left(\frac{T}{3.3\,\mathrm{eV}}\right)^2 \frac{\Omega_B h^2}{0.02}, \\ \frac{\Gamma_{\mathrm{Compton},e}}{H} &\simeq 1.3 \times 10^6 \left(\Omega_0 h^2\right)^{-1/2} \left(\frac{T_\gamma}{1\,\mathrm{eV}}\right)^{5/2} \frac{T_e}{T_\gamma}, \\ &\simeq 8.1 \left(\Omega_0 h^2\right)^{-1/2} \left(\frac{T_\gamma}{100\,\mathrm{K}}\right)^{5/2} \frac{T_e}{T_\gamma}. \end{aligned}$$

And so we see that the electron mean free time is quite short, but the photon mean free time is becoming comparable to the Hubble time at temperatures below about 3 eV.

Note that there is a further issue with Compton scattering. Energy transfer allows photons to be moved from one energy bin to another. But there is no change in the total photon density. Therefore, if the photon distribution happened to have a number density inappropriate for a thermal distribution at the given temperature, then the photon distribution could never reach a black-body by the action of Compton scattering alone. This will be relevant for us later. In the interim we will assume that the number density is correct, which is consistent with the photon distribution starting as thermal at some higher temperature.

4.4. **Free-Free Processes.** Free-free processes are bremsstrahlung and the inverse absorption process. Free-free processes require a spectator body, such as a proton, to get the necessary acceleration,

$$p + e \longleftrightarrow p + e + \gamma.$$

So the rate is proportional to the square of the ionization density. Free-free processes would require a lecture of their own, and the results have a strong frequency dependence which makes them difficult to use. For example, see [Longair, vol. 1, p. 71].

We will restrict ourselves to two observations. First, at least for frequencies near the temperature, $w \simeq T$, free-free processes in the primordial plasma can effectively process photons down to temperatures of about 1 eV. So these processes may be relevant even after Compton scattering of photons has stopped being effective. Second, free-free processes obviously effect both photon number and energy, so they can be effective at thermalizing radiation.

4.5. **Recombination.** Finally we come to the main player, the recombination reaction itself,

$$p + e \longleftrightarrow H + \gamma.$$

For the moment we consider only recombination to the ground state.

$$\langle \sigma_r v \rangle = \frac{4\pi^2 \alpha_{em}}{m_e^2} \frac{B_H}{(3m_e T)^{1/2}} \simeq 4.7 \times 10^{-24}\,\mathrm{cm}^2 \left(\frac{1\,\mathrm{eV}}{T} \right)^{1/2}.$$

The rate is $\Gamma \simeq x\eta n_\gamma \langle \sigma_r v \rangle$. But now things get a little complicated. We must track the ionization fraction through decoupling, where it changes rapidly. We expect x to drop rapidly and then freeze-out at some residual value, and we would like to calculate this value as well as understand the decoupling of radiation. We will do this in the next section.

5. Decoupling and Recombination

5.1. **The Robust Black-Body.** We now have enough information to understand what effect decoupling has on the radiation spectrum. From the above we see that Compton scattering of photons is becoming ineffective at temperatures of about 3 eV. But Compton scattering of electrons is effective until very late times. This is a result of the very small value of η. Therefore, the electron temperature is locked to the radiation temperature throughout all of decoupling and recombination. By our previous argument, this implies that the black-body spectrum cannot be distorted by the changing temperature and density of the charged particles. Even if every electron is involved in scattering (and they are, based on the short electron mean free time), only a small fraction of photons can be scattered out of the thermal black-body. If one thinks of the photons as a heat bath, then the change in their distribution is a finite-size effect, of order $\eta \simeq 10^{-9}$.

So it seems that the black-body is immune to scattering effects during decoupling and recombination. This is actually not quite true. We will see later how a distortion at high frequency can arise from hydrogen recombination.

In the next section we calculate the residual ionization fraction using the simplest dynamic calculation of recombination. This is also the last step that we need to verify our picture of decoupling.

5.2. **Calculating the Residual Ionization Dynamically.** Thermal estimates for the ionization fraction were useful to us in understanding decoupling. But they are completely inadequate for calculating the residual ionization, which depends on the dynamical freeze-out of the charged particles. Furthermore, we still need to check the validity of the recombination temperature that we calculated using the Saha equation.

There are a few different ways to do this. We will pick the route which is most illustrative of the general freeze-out calculation. We begin with a kinetic equation for the evolution of the ionization fraction.

$$\dot{n}_e + 3\frac{\dot{a}}{a}n_e = -\langle\sigma_r v\rangle(t)\left[n_e^2 - (n_e^0)^2\right],$$

where n_e^0 is the density in thermal equilibrium. Let $s = (1\,\mathrm{eV}/T)$ and write $n_e = xn_B$. Then the above becomes

$$\frac{dx}{ds} = -\frac{b}{s^2}\left[x^2 - (x^0(s))^2\right],$$

where $b \simeq 1.4 \times 10^5(\Omega_B h^2)(\Omega_0 h^2)^{-1/2}$. We have assumed matter domination, which is the origin of the factor of $\Omega_0 h^2$. The equilibrium ionization is the solution of the Saha equation

$$\frac{(x^0(s))^2}{1-x^0(s)} = \frac{1}{n_B}\left(\frac{m_e T}{2\pi}\right)^{3/2} \exp\left(-\frac{B_H}{T}\right),$$
$$= 9.4\times 10^7 \eta^{-1} s^{3/2} \exp\left(-\frac{B_H}{1\,\mathrm{eV}} s\right).$$

The deep past corresponds to $s \to 0$, and the future to $s \to \infty$. We guess from experience that there will be a sharp transition in x at some intermediate value of s, followed by a gradual relaxation to a constant value, the residual ionization.

For those who know about such things, this sort of behaviour is similar to the behaviour of boundary layer solutions. As for boundary layers, it can be handled by matching an interior and exterior solution. The interior solution is constructed with a singular perturbation expansion.

First we obtain the behaviour in the interior or transition region. Let $\Delta = x - x^0(s)$. Then write the equation inthe form

$$\epsilon\frac{d\Delta}{ds} = -\frac{dx^0}{ds} - \frac{b}{s^2}\Delta(\Delta + 2x^0(s)).$$

We have inserted an ϵ on the derivative. The singular perturbation expansion is asymptotic as $\epsilon \to 0$. The leading term is simply

$$\frac{b}{s^2}\Delta(\Delta + 2x^0(s)) \sim -\frac{dx^0}{ds}.$$

Δ is much smaller than x^0 as we move backward in time, and this is consistent with the solution that we will determine. So we write

$$\Delta \sim -\frac{s^2}{2b}\frac{d}{ds}\ln x^0(s), \quad s \to 0.$$

The derivative of x^0 comes almost entirely from the exponential term, so

$$-\frac{d}{ds}\ln x^0 \simeq \frac{1}{2}\left(\frac{B_H}{1\,\mathrm{eV}}\right) \simeq 6.8.$$

Therefore the solution in the region of rapid variation is given by

$$\Delta \sim 3.4\frac{s^2}{b}, \quad s \to 0.$$

For large s, the exterior region, we expect $x \gg x^0$. Therefore we write

$$\frac{dx}{ds} = -\frac{b}{s^2}x^2(1 + \mathcal{O}((x^0/x)^2)).$$

Integrating from some s_* to ∞, we get

$$\frac{1}{x_\infty} - \frac{1}{x(s_*)} = \frac{b}{s_*}.$$

The point s_* must be chosen in the exterior region, but early enough that $x(s_*)^{-1}$ is negligible. Then the residual ionization will be

$$x_\infty \simeq \frac{s_*}{b}.$$

We choose s_* in the matching region, where the interior and exterior solutions overlap. That such a region exists is guaranteed by the theory of these expansions. Choose s_* to be where $\Delta \simeq x^0(s)$, so that the deviation from thermal is becoming of order unity. Therefore

$$\frac{3.4}{b} s_*^2 \simeq x^0(s_*).$$

Some tedious arithmetic gives

$$s_* \simeq 3.6 - 0.074 \ln \frac{\Omega_B}{\Omega_0}.$$

Now we have everything we need. The following table summarizes some numbers.

Ω_B/Ω_0	s_*	T_* (eV)	$x_\infty(\Omega_B h^2)(\Omega_0 h^2)^{-1/2}$
0.05	3.82	0.262	
0.10	3.77	0.265	2.7×10^{-5}
0.15	3.74	0.267	

Recall the equilibrium estimate obtained from the Saha equation, $T_* \simeq 0.3\,\mathrm{eV}$. This turned out to be not such a bad estimate. Therefore simple recombination can be reasonably estimated by equilibrium arguments. The non-equilibrium analysis also gives us the residual ionization of the Universe, $x_\infty \approx 10^{-4}$. Notice that this ionization fraction is enough to keep the charged particles thermal by Coulomb scattering. So we predict a primordial relic of thermal ionized particles. Also, we know that these ionized particles track the radiation temperature until very late times, say $z \simeq 50$, by our previous estimate of the electron mean free time.

5.3. **Transparency of the Universe.** We have one last thing to check, which is the assertion that the Universe is essentially transparent after decoupling and recombination. Recall the time-scales for Thomson and Coulomb scattering of photons,

$$t_{\mathrm{T},\gamma} \simeq 3.1 \times 10^9 \,\mathrm{sec}\, \frac{1}{x} \left(\frac{1\,\mathrm{eV}}{T}\right)^3 \frac{\Omega_B h^2}{0.02},$$
$$t_{\mathrm{Compton},\gamma} \simeq 1.6 \times 10^{15} \,\mathrm{sec}\, \frac{1}{x} \left(\frac{1\,\mathrm{eV}}{T}\right)^3 \frac{\Omega_B h^2}{0.02}.$$

The best thing to do is to integrate the scattering probability to get the optical depth. If we write each of the above in the form $t_{\mathrm{X}} = c_X x^{-1} \left(1\,\mathrm{eV}/T\right)^3$, then the optical depth since decoupling is

$$\tau = \int_{t_{\mathrm{dec}}}^{t_{\mathrm{now}}} dt\, t_{\mathrm{X}}^{-1}.$$

At this point we will assume that the Universe remains matter dominated for some reasonable fraction of the time from decoupling until now, just so we can use the definite formula $t \simeq H^{-1} \simeq 1.1{\times}10^{12}\,\mathrm{sec}\,(1\,\mathrm{eV}/T)^{3/2}(\Omega_0 h^2)^{-1/2}$. Then we have

$$\tau_X \simeq \frac{3}{2} x \frac{1.1 \times 10^{12}\,\mathrm{sec}}{c_X} \int_{T_{\mathrm{now}}}^{T_{\mathrm{dec}}} \left(\frac{T}{1\,\mathrm{eV}}\right)^{3/2} \frac{dT}{T}$$
$$\simeq x \left(\frac{1.1 \times 10^{12}\,\mathrm{sec}}{c_X}\right) \left(\frac{T_{\mathrm{dec}}}{1\,\mathrm{eV}}\right)^{3/2}.$$

Taking the decoupling temperature to be 0.2 eV and using the relic ionization which we calculated, $x \approx 10^{-4}$, we get an optical depth for Thomson scattering of $\tau_T \approx 3 \times 10^{-3}$. The optical depth to Compton scattering is clearly very small. So the Universe is indeed transparent after decoupling, at least until low redshifts where UV radiation from star formation will begin increasing the ionization fraction. Also, some photons will travel by chance through isolated hot regions of the current Universe, such as galaxy clusters, so one should expect some scattering off of ionized gas in such regions. Compton scattering of photons in such regions is called the Sunyaev-Zeldovich effect. At some point later in our microwave background discussions we will examine this effect further.

5.4. **The Truth About Hydrogen Production.** Have we now finished with decoupling and recombination? As one might guess, we have only begun. The calculation that was made can only be considered a simplified picture.

In fact, I have lied to you. Recombination to neutral hydrogen turns out to be tricky. The problem is as follows. Early in recombination things proceed as we have outlined and calculated. But as the recombined fraction grows, a population of 13.6 eV photons grows with it. These photons cannot be thermalized at all. Even Compton scattering is inoperative at these temperatures below 1 eV, so the photons cannot even be redistributed across the spectrum. So recombination is dumping a population of photons in the frequency region corresponding to Lyman transitions. Eventually these photons have a density comparable to hydrogen and they begin to hold up recombination to the ground state or to excited states connected by Lyman transitions.

This seems like a real problem. How can we fix it? There are two important effects to be considered.

- The product photons will eventually redshift out of the 13.6 eV resonance region. So some recombination to the ground state will continue, at a reduced rate.
- Recombination to the $2s$ state will also be occuring. Other similar states are slightly relevant, but the $2s$ is most important. This is a metastable state, which decays by a 2-photon transition, with a rate $\Gamma_{2s\to 1s} \simeq 8.23\,\mathrm{sec}^{-1}$, compared to $\Gamma_{2p\to 1s} \simeq 6.25 \times 10^8\,\mathrm{sec}^{-1}$.

Of these effects, the most important is the second. So recombination is held up waiting for out-of-equilibrium decays of the $2s$ state. But it eventually finishes since we have thousands of years for the process to complete. This hardly changes the predicted recombination temperature but it does change the prediction for the residual ionization, decreasing it by about a factor of 3.

This non-equilibrium recombination process leaves a pile of emitted photons, smeared over a region of the spectrum near Lyman frequencies and a little below. This region corresponds to frequencies $\nu \approx 1000\,\mathrm{GHz}$ today. Unfortunately, this high frequency part of the spectrum will almost certainly never be measured since it is swamped by dust emission in the Galaxy.

To do the calculation of non-equilibrium recombination including the $2s$ state, or for that matter including other states and states of Helium as well, requires some detailed work. For some details on the hydrogen $2s$ calculation you can see [Peebles, Principles of Physical Cosmology, p. 167].

6. Phenomenology of Spectral Distortions

We have seen that recombination itself creates a high frequency spectral distortion, which will probably never be observed. What about other possible energy releases? What sort of distortions might they produce?

First, recall that at very high temperatures the radiation spectrum will be thermalized by processes involving changes in photon number and energy.

These include free-free processes and processes such as $e+\gamma \longrightarrow e+2\gamma$, the so-called double Compton process. These processes effectively thermalize radiation down to a temperature of about 1 keV. Therefore, no energy release at temperatures higher than this can leave an imprint on the radiation spectrum.

At temperatures below 1 keV, Compton scattering is still operating. But as we have seen, Compton scattering cannot produce a black-body spectrum if the number density of photons is not appropriate. Therefore an energy release into the radiation background at this time would lead to a so-called chemical potential distortion. This means that the spectrum would reach statistical equilibrium with charged particles, but would have a nonzero chemical potential, just like a relativistic species with conserved particle number would have. This distortion is measured in terms of the dimensionless chemical potential, $\mu_0 = \mu/T$.

At temperatures low enough that Compton scattering is not operating, energy releases into the radiation apectrum are not thermalized, so one would see their imprint directly.

More to the point are late time scattering events. For example, background photons traversing a region of ionized gas at late times will experience Compton up-scattering, called ”Comptonization”. Comptonization transfers photons from the Rayleigh-Jeans region to the Wien region. This means that the Rayleigh-Jeans temperature (”antenna temperature”) will be lower than that extracted from the Wien region. The temperature decrement in the Rayleigh-Jeans region is parametrized by a dimensionless parameter y,

$$\Delta T_{RJ} = -2yT_\gamma,$$

where y is determined from an integral along the photon path

$$\begin{aligned} y &= \int dt\, \sigma_{\rm Compton} n_e \left(\frac{T_e - T_\gamma}{T_e}\right) \\ &= \int dt\, \sigma_T n_e \left(\frac{T_e - T_\gamma}{m_e}\right). \end{aligned}$$

This formula has a simple interpretation. If you think of the integrand as dT_γ/dt, then we see that the photons have a heating rate which is proportional to the temperature difference, with rate constant set by the microscopic scattering cross section; this is basically just the Newton-Fourier law of cooling. We expect that such a distortion must be produced at some level by a hot intergalactic medium. When photons pass through a hot region such as a cluster of galaxies, the heating is called the Sunyaev-Zeldovich effect. As we have discussed, it is not really a thermal heating process because Compton scattering cannot equilibrate photon number. So this scattering shifts photons from the Rayleigh-Jeans side of the spectrum to the Wien side.

7. OBSERVATIONS

The Dicke radiometer, which is a differential temperature measurement device, made possible a few early experiments on the radiation background. In 1946 Dicke took data at $6\,\mathrm{cm} \simeq 20\,\mathrm{GHz}$ which gave a limit $T_\gamma < 20\,\mathrm{K}$. This temperature limit and similar measurements at a single frequency are "antenna temperature" limits. The antenna temperature is the temperature which one would assign, given an intensity, assuming the measurement occurs on the Rayleigh-Jeans side of a black-body spectrum.

There is an interesting history of other early observations, some of which should probably qualify as detections, although cosmological interpretations were apparently not assigned to them. For instance, Andrew McKellar in 1940 and Walter Adams in 1941 made spectral absorption line measurements of rotational states of interstellar CN which were consistent with thermal equilibrium at 2.73 K. The lines are at roughly $\nu \approx 100\,\mathrm{GHz}$ and $\nu \approx 200\,\mathrm{GHz}$, although I believe the early measurements were done only at 100 GHz, which corresponds to the first excited rotational state.

But it was not until the early 1960's that a group of people began to seriously consider a cosmological background, as it became clear that such a background could be observable and that this would be an important test of the hot big bang model. Also, Dicke was interested in measurements to test Brans-Dicke cosmology, so work began on an apparatus in Princeton around 1964.

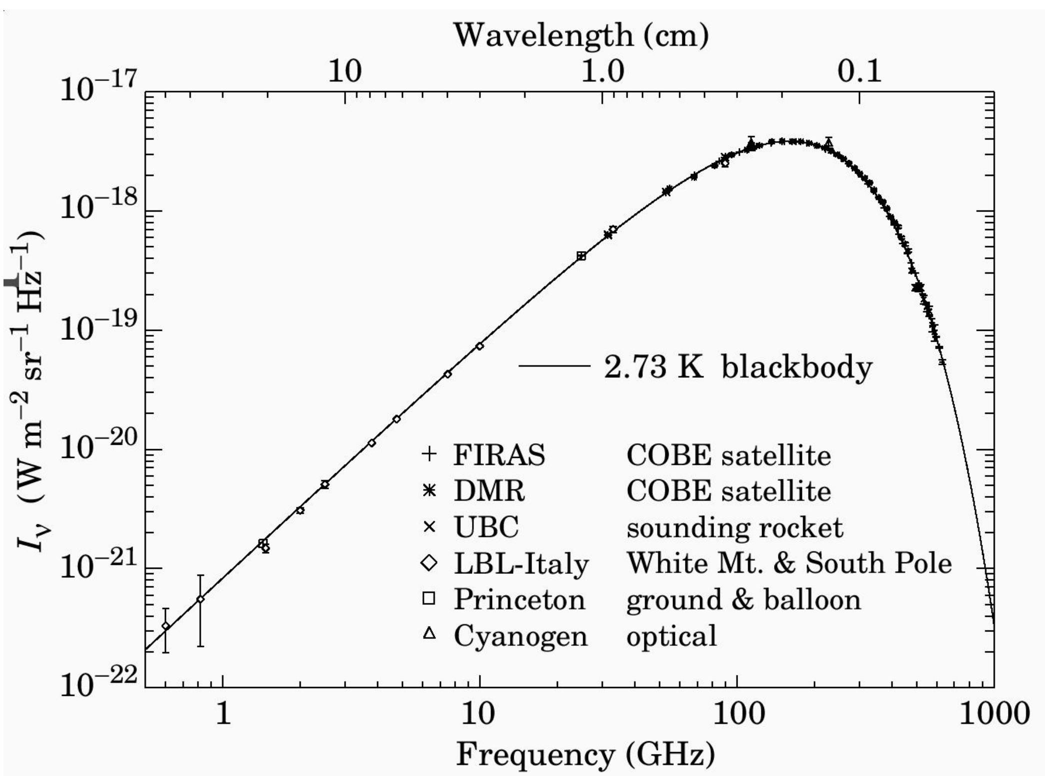
Meanwhile, in Holmdel New Jersey, a 7 cm microwave horn (also a Dicke radiometer) initially intended for satellite communications continued to show an excess antenna noise, as reported by E. Ohm in 1961 in a Bell technical report. In 1965 Penzias and Wilson were trying to use the same telescope for radio-astronomy observations and also found this noise. But now the time was right, and after discussions with Dicke's group the true nature of this discovery was realized. Two back-to-back papers appeared in Ap. J. (**142**, 414 and 419). The first, by the Dicke group, explained the significance of the measurement, and the second was the Penzias and Wilson "excess antenna temperature" paper.

After the initial measurement there was a long history of successive refinements, and the search for fluctuations was on almost immediately. We will have more to say about fluctuations in later lectures.

The current state of the art for spectral measurements is summarized by the following data and limits (Smoot and Scott, 1997).

$$\begin{aligned} T_\gamma &= 2.728 \pm 0.002\,\mathrm{K}, \\ |y| &< 1.2 \times 10^{-5} (95\% CL), \\ |\mu_0| &< 9 \times 10^{-5} (95\% CL), \end{aligned}$$

These represent a global average. The COBE FIRAS temperature itself is $T = 2.728 \pm 0.004\,\mathrm{K}$. The temperature and μ distortion numbers are apparently dominated by the COBE FIRAS analysis. This is not to say that other measurements are not still important. COBE FIRAS covered the frequency range 68 GHz – 640 GHz, but it is important to constrain lower frequency distortions as well.



The measurements at low and high frequencies are hindered by foreground signals. At low frequencies there is synchrotron and free-free emission from ionized regions. At high frequencies there is infrared dust emission from the Galaxy. Typically one models and then subtracts Galactic emission, but clearly there is a fundamental limitation to this approach and therefore a fundamental limitation to sensitivity at these frequencies.

As a final note, it seems that the location of the primordial background spectrum is fortuitous. A few decades to the right or left and it might well have been obscured by foreground emissions. But we have become used to the occasional coincidence in cosmology.

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